Applied Mathematics Letters 98 (2019) 49–56

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Uniform convergence of V-cycle multigrid finite element method for one-dimensional time-dependent fractional problem

Minghua Chen [a,*], Wenya Qi [a], Yantao Wang [b]

[a] *School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, PR China*
[b] *Department of Mathematics, Southern University of Science and Technology, Shenzhen 518055, PR China*

A R T I C L E I N F O

Article history:
Received 9 April 2019
Received in revised form 24 May 2019
Accepted 24 May 2019
Available online 30 May 2019

Keywords:
Uniform convergence of V-cycle multigrid method
Fractional τ-norm
Fast Fourier transform

A B S T R A C T

Analysing the fractional τ-norm, the uniform convergence of the V-cycle multigrid FEM for the time-dependent fractional problem is strictly proved when $\tau \to 0$. The numerical experiments are performed to verify the convergence with $\mathcal{O}(N\log N)$ complexity by fast Fourier transform method.
© 2019 Elsevier Ltd. All rights reserved.

1. Introduction

In this paper we study the V-cycle multigrid FEM for solving the time-dependent fractional problem whose prototype is [1], for $1 < \alpha < 2$,

$$\frac{\partial u}{\partial t} - \nabla_x^\alpha u(x,t) = f(x,t) \quad \text{in} \quad \Omega \times (0,T] \tag{1.1}$$

with the initial condition $u(x,0) = u_0(x)$, $x \in \Omega = (x_L, x_R)$ and the homogeneous Dirichlet boundary conditions. The fractional derivative is defined by [2,3]

$$\nabla_x^\alpha u(x,t) = \kappa_\alpha \left[{}_{x_L}D_x^\alpha + {}_xD_{x_R}^\alpha \right] u(x,t), \quad \kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0.$$

When considering iterative solvers for the large-scale linear systems arising from the approximation of elliptic partial differential equations (PDEs), multigrid methods (MGM) are often optimal order process [4,5]. The elegant theoretical framework and uniform convergence of V-cycle MGM for second order elliptic

* Corresponding author.
E-mail address: chenmh@lzu.edu.cn (M. Chen).

https://doi.org/10.1016/j.aml.2019.05.039
0893-9659/© 2019 Elsevier Ltd. All rights reserved.

equation is well established in [5,6]. The convergence rate independent of the number of levels is presented by multigrid FEM for elliptic equations with variable coefficients [7]. In the case of multilevel matrix algebras with special prolongation operators, the convergence rate of the V-cycle MGM is derived in [8] for the elliptic PDEs. Using the traditional (simple) prolongation operator, for the time-dependent second elliptic problems, the new convergence proofs for V-cycle MGM including multilevel linear systems are given in [9]. For the time-independent fractional PDEs, based on the idea of [5,10], the convergence rate of the V-cycle MGM is discussed in [11–13] and the nearly uniform convergence result is derived in [14]. For the time-dependent fractional PDEs, the convergence rate of the two-grid method has been performed in [1,15] by following the ideas in [16]; and the convergence of the V-cycle MGM is investigated with a fixed time step $\tau > 0$ [17].

However, for $\tau \to 0$, as far as we know, the convergence rate of the V-cycle multigrid FEM has not been considered for a time-dependent PDEs. In this paper, based on introducing and analysing the fractional τ-norm, the convergence rate of the V-cycle MGM is strictly proved. Moreover, the fast Toeplitz matrix–vector multiplication is utilized to lower the computational cost with only $\mathcal{O}(N\log N)$ complexity by fast Fourier transform (FFT) method [15,18], where N is the number of the grid points.

The outline of the paper is as follows. In the next section, we briefly review the full discretization scheme of the time-dependent problem (1.1). In Section 3, we first define the fractional τ-norm and prove the convergence estimates of the V-cycle MGM with time-dependent fractional PDEs. The numerical experiments are reported in Section 4. Finally, we conclude the paper with some remarks.

2. Preliminaries

Define the bilinear form b: $H_0^{\alpha/2}(\Omega) \times H_0^{\alpha/2}(\Omega) \to \mathbb{R}$ as [2]

$$b(u, v) = -2\kappa_\alpha \left({}_{x_L}D_x^{\alpha/2} u, {}_x D_{x_R}^{\alpha/2} v \right). \tag{2.1}$$

Let V_k denote C^0 piecewise linear functions with the uniform meshes $h_k = \frac{1}{2} h_{k-1}$, i.e. $V_{k-1} \subset V_k,\ k \geq 1$, and $t_n = n\tau,\ n = 0, 1, \ldots, N,\ \tau = \frac{T}{N}$ is time step. Then the full-discretizaion problems with the Crank–Nicolson scheme in time direction is: Find $u_k^n \in V_k$ such that

$$a_\tau(u_k^n, v) = (g^{n-1}, v) \quad \forall v \in V_k, \tag{2.2}$$

where $(g^{n-1}, v) = \tau^{-1}(u_k^{n-1}, v) - \frac{1}{2} b(u_k^{n-1}, v) + (f_k^{n-1/2}, v)$, and

$$a_\tau(w, v) = \tau^{-1}(w, v) + \frac{1}{2} b(w, v), \quad v, w \in V_k. \tag{2.3}$$

The operator $A_{k,\tau} : V_k \to V_k$ and $g_k^{n-1} : V_k \to V_k$ are defined by

$$(A_{k,\tau} w, v)_k = a_\tau(w, v), \quad (g_k^{n-1}, v)_k = (g^{n-1}, v) \quad \forall v, w \in V_k. \tag{2.4}$$

Here the mesh-dependent inner product is defined by [10]

$$(w, v)_k := h_k \sum_{i=1}^{n_k} w(p_i) v(p_i), v, w \in V_k,$$

and $\{p_i\}_{i=1}^{n_k}$ is the set of internal vertices.

From (2.2) and (2.4), we obtain

$$A_{k,\tau} z = g,\ g := g_k^{n-1} \in V_k, \quad z := u_k^n \in V_k. \tag{2.5}$$

Since $A_{k,\tau}$ is symmetric positive definite with respect to $(\cdot, \cdot)_k$, we can define a scale of mesh-dependent norms $\|\!|\cdot|\!\|_{s,k,\tau}$ in the following way

$$|\!|\!| v |\!|\!|_{s,k,\tau} := \sqrt{(A_{k,\tau}^s v, v)_k}. \tag{2.6}$$

Lemma 2.1 ([2]). *The bilinear form $b(\cdot,\cdot)$ is coercive and continuous on $H_0^{\alpha/2}(\Omega)\times H_0^{\alpha/2}(\Omega)$ with $1<\alpha<2$, i.e. there exists a constant such that*

$$b(u,u)\geq C_0\|u\|^2_{H_0^{\alpha/2}(\Omega)},\quad |b(u,v)|\leq C_1\|u\|_{H_0^{\alpha/2}(\Omega)}\|v\|_{H_0^{\alpha/2}(\Omega)}.$$

3. Uniform convergence of V-cycle multigrid FEM for (2.2)

The time-dependent fractional MGM can be treated as the elliptic equations arising at a fixed time step $\tau>0$ [17]. However, the bilinear form $a_\tau(w,v)$, see (2.3), is unbounded in the traditional norm when the time step $\tau\to 0$. To overcome this gap, we below introduce the fractional τ-norm.

Definition 3.1. Let $P_k : H_0^{\alpha/2}(\Omega)\to V_k$ be the orthogonal projection with respect to $a_\tau(\cdot,\cdot)$, i.e.

$$a_\tau(v,w)=a_\tau(P_kv,w)\quad \forall w\in V_k. \tag{3.1}$$

Let K_k be the iteration matrix of the smoothing operator. Here, we take K_k to be the weighted (damped) Jacobi iteration matrix

$$K_k=I-S_kA_{k,\tau},\quad S_k:=S_{k,\eta}=\eta D_{k,\tau}^{-1} \tag{3.2}$$

with a weighting factor $\eta\in(0,1/2]$, and $D_{k,\tau}$ is the diagonal of $A_{k,\tau}$. A multigrid process can be regarded as defining a sequence of operators $B_k : V_k\mapsto V_k$ which is an approximate inverse of $A_{k,\tau}$ in the sense that $\|I-B_kA_{k,\tau}\|$ is bounded away from one [9]. The V-cycle multigrid algorithm [5,10] is provided in Algorithm 1.

Algorithm 1 V-cycle Multigrid Algorithm: Define $B_1=A_{1,\tau}^{-1}$. Assume that $B_{k-1} : V_{k-1}\mapsto V_{k-1}$ is defined. We shall now define $B_k : V_k\mapsto V_k$ as an approximate iterative solver for the equation $A_{k,\tau}z=g$.

1: Pre-smooth: Let $S_{k,\eta}$ be defined by (3.2), $z_0=0$, $l=1:m_1$, and $z_l=z_{l-1}+S_{k,\eta_{pre}}(g_k-A_{k,\tau}z_{l-1})$.
2: Coarse grid correction: Denote $e^{k-1}\in V_{k-1}$ as the approximate solution of the residual equation $A_{k-1}e=I_k^{k-1}(g-A_{k,\tau}z_{m_1})$ with the iterator B_{k-1}: $e^{k-1}=B_{k-1}I_k^{k-1}(g-A_{k,\tau}z_{m_1})$.
3: Post-smooth: $z_{m_1+1}=z_{m_1}+I_{k-1}^k e^{k-1}$, $l=m_1+2:m_1+m_2+1$, and $z_l=z_{l-1}+S_{k,\eta_{post}}(g-A_{k,\tau}z_{l-1})$.
4: Define $\mathrm{MG}(k,z_0,g):=B_kg=z_{m_1+m_2+1}$.

Based on the (2.3), we define the fractional τ-norm

$$\|v\|^2_{\tau,\alpha}=\tau^{-1}\|v\|^2_{L^2(\Omega)}+\|v\|^2_{H^\alpha(\Omega)}\quad \forall v\in H^\alpha(\Omega). \tag{3.3}$$

In order to estimate the spectral radius, $\rho(A_{k,\tau})$, of $A_{k,\tau}$, we first introduce the following lemmas.

Lemma 3.1. *The bilinear form $a_\tau(u,v)$ is symmetrical, continuous and coercive. In other words, there exist two positive constants C_2, C_3 such that*

$$a_\tau(u,u)\geq C_2\|u\|^2_{\tau,\alpha/2}\quad\text{and}\quad |a_\tau(u,v)|\leq C_3\|u\|_{\tau,\alpha/2}\|v\|_{\tau,\alpha/2}.$$

Proof. According to (2.3) and Lemma 2.1, there exists

$$a_\tau(u,u)=\tau^{-1}(u,u)+\frac{1}{2}b(u,u)\geq\tau^{-1}(u,u)+\frac{C_0}{2}\|u\|^2_{H^{\alpha/2}(\Omega)}\geq C_2\|u\|^2_{\tau,\alpha/2}$$

with $C_2 = \min\{1, C_0/2\}$. On the other hand, using Lemma 2.1, we have

$$\begin{aligned}
|a_\tau(u,v)| &\le \tau^{-1}|(u,v)| + \frac{1}{2}|b(u,v)| \le \left(1+\frac{1}{2}C_1\right)\left(\tau^{-1}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|u\|_{H^{\alpha/2}(\Omega)}\|v\|_{H^{\alpha/2}(\Omega)}\right)\\
&\le \left(1+\frac{1}{2}C_1\right)\Big\{\left(\tau^{-2}\|u\|^2_{L^2(\Omega)}\|v\|^2_{L^2(\Omega)} + \|u\|^2_{H^{\alpha/2}(\Omega)}\|v\|^2_{H^{\alpha/2}(\Omega)}\right)\\
&\quad +\tau^{-1}\|u\|^2_{L^2(\Omega)}\|v\|^2_{H^{\alpha/2}(\Omega)} + \tau^{-1}\|v\|^2_{L^2(\Omega)}\|u\|^2_{H^{\alpha/2}(\Omega)}\Big\}^{1/2}\\
&= \left(1+\frac{1}{2}C_1\right)\|u\|_{\tau,\alpha/2}\|v\|_{\tau,\alpha/2}.
\end{aligned}$$

The proof is completed. □

According to (2.6), (3.3) and Lemma 3.1, it is easy to get

$$\begin{aligned}
c\|v\|_{L^2(\Omega)} &\le |||v|||_{0,k,\tau} \le C\|v\|_{L^2(\Omega)},\\
c\|v\|_{\tau,\alpha/2} &\le |||v|||_{1,k,\tau} \le C\|v\|_{\tau,\alpha/2},\\
c\|A_{k,\tau}v\|_{L^2(\Omega)} &\le |||v|||_{2,k,\tau} \le C\|A_{k,\tau}v\|_{L^2(\Omega)}.
\end{aligned} \tag{3.4}$$

Lemma 3.2 (*[19]*)**.** *Let $s_1 < s_2$ be two real numbers, and $\mu = (1-\theta)s_1 + \theta s_2$ with $0 \le \theta \le 1$. Then there exists a constant such that $\|v\|_\mu \le C\|v\|^{1-\theta}_{s_1}\|v\|^{\theta}_{s_2} \quad \forall v \in H^{s_2}(\Omega)$.*

Lemma 3.3. *Let $A_{k,\tau}$ be defined by (2.4). Then there exists a constant such that*

$$\rho(A_{k,\tau}) \le C(1+\tau^{-1}h^\alpha)h^{-\alpha}.$$

Proof. From Lemmas 2.1, 3.2 and inverse estimation of [19], there exists

$$\begin{aligned}
b(v,v) &\le C_1\|v\|^2_{H^{\alpha/2}(\Omega)} \le C_1\left(C_2\|v\|^{1-\alpha/2}_{L^2(\Omega)}\cdot\|v\|^{\alpha/2}_{H^1(\Omega)}\right)^2\\
&\le C_1\left(C_2\|v\|^{1-\alpha/2}_{L^2(\Omega)}\cdot h^{-\alpha/2}\|v\|^{\alpha/2}_{L^2(\Omega)}\right)^2 \le C_3h^{-\alpha}\|v\|^2_{L^2(\Omega)}.
\end{aligned}$$

Let Λ be an eigenvalue of $A_{k,\tau}$ with eigenvector v. From the above equation, (3.4) and Lemmas 3.1, 2.1, we have

$$\Lambda(A_{k,\tau}) = \frac{(A_{k,\tau}v,v)_k}{(v,v)_k} = \frac{a_\tau(v,v)}{(v,v)_k} \le \frac{C_4\|v\|^2_{\tau,\alpha/2}}{\|v\|^2_{L^2(\Omega)}} \le \frac{C_5\left(\tau^{-1}\|v\|^2_{L^2(\Omega)} + b(v,v)\right)}{\|v\|^2_{L^2(\Omega)}} \le C(1+\tau^{-1}h^\alpha)h^{-\alpha}.$$

The proof is completed. □

Lemma 3.4. *Let $A_{k,\tau} = \{a_{i,j}\}^{n_k}_{i,j=1}$ be defined by (2.5). Then*

$$\frac{\eta}{\rho(A_{k,\tau})}(\nu_k,\nu_k) \le (S_k\nu_k,\nu_k) \le (A^{-1}_{k,\tau}\nu_k,\nu_k) \quad \forall \nu_k \in V_k,$$

where $S_k = \eta D^{-1}_{k,\tau}$, $\eta \in (0,1/2]$ and $D_{k,\tau}$ is the diagonal of $A_{k,\tau}$.

Proof. It is easy to check that $A_{k,\tau}$ is a weakly diagonally dominant symmetric Toeplitz M-matrix [1,20], i.e., $A_{k,\tau}$ is a positive definite matrix with positive entries on the diagonal and nonpositive off-diagonal entries and the diagonal element of a matrix is at least as large as the sum of the off-diagonal elements in the same row or column [18]. Then the similar arguments can be performed as Lemma 2.4 of [9], the desired result is obtained. □

Remark 3.1. We conclude that, for the fractional problem (1.1), the stiffness matrix of the linear finite element approximation on a uniform grid, after proper scaling, is equivalent to the one obtained by the finite difference scheme.

Lemma 3.5. *For any real number θ, it holds*

$$|a_\tau(v,w)| \le |||v|||_{1+\theta,k,\tau} |||w|||_{1-\theta,k,\tau} \quad \forall v,w \in V_k.$$

Proof. Let λ_i with $1 \le i \le n_k$ be the eigenvalues of the operator $A_{k,\tau}$ and ψ_i be the corresponding eigenfunction satisfying the orthogonal relation $(\psi_i,\psi_j)_k = \delta_{i,j}$. We can write $v = \sum_{i=1}^{n_k} c_i\psi_i, w = \sum_{j=1}^{n_k} d_j\psi_j$. From (2.4) and (2.6), we obtain

$$\begin{aligned} a_\tau(v,w) = (A_{k,\tau}v,w)_k &= \left(\sum_{i=1}^{n_k}\lambda_i c_i\psi_i, \sum_{j=1}^{n_k} d_j\psi_j\right)_k = \sum_{i=1}^{n_k}\lambda_i c_i d_i \le \left(\sum_{i=1}^{n_k} c_i^2\lambda_i^{1+\theta}\right)^{1/2}\left(\sum_{i=1}^{n_k} d_i^2\lambda_i^{1-\theta}\right)^{1/2} \\ &= \left(A_{k,\tau}^{1+\theta}v,v\right)_k^{1/2}\left(A_{k,\tau}^{1-\theta}w,w\right)_k^{1/2} = |||v|||_{1+\theta,k,\tau}|||w|||_{1-\theta,k,\tau}. \end{aligned}$$

The proof is completed. □

Lemma 3.6. *For $v \in H_0^{\alpha/2}(\Omega)$, there exists a positive constant C such that*

$$\|(I-P_{k-1})v\|_{L^2(\Omega)} \le C\|(I-P_{k-1})v\|_{\tau,\alpha/2}\left(\sup_{\varphi\neq 0}\left\{\frac{1}{\|\varphi\|_{L^2(\Omega)}}\inf_{v_{k-1}\in V_{k-1}}\|w_\varphi - v_{k-1}\|_{\tau,\alpha/2}\right\}\right),$$

where $w_\varphi \in H_0^{\alpha/2}(\Omega)$ is the unique solution of $a_\tau(\nu,w_\varphi) = (\varphi,\nu)$ $\forall \nu \in H_0^{\alpha/2}(\Omega)$.

In particular, if $w_\varphi \in H^\alpha(\Omega)$, we have

$$\|(I-P_{k-1})v\|_{L^2(\Omega)} \le Ch^{\alpha/2}\left(1+\tau^{-1}h^\alpha\right)^{1/2}\|(I-P_{k-1})v\|_{\tau,\alpha/2}.$$

Proof. For $v_{k-1} \in V_{k-1}$, we have

$$\begin{aligned} \|(I-P_{k-1})v\|_{L^2(\Omega)} &= \sup_{\varphi\neq 0}\frac{|(\varphi,(I-P_{k-1})v)|}{\|\varphi\|_{L^2(\Omega)}} = \sup_{\varphi\neq 0}\frac{|a_\tau((I-P_{k-1})v,w_\varphi)|}{\|\varphi\|_{L^2(\Omega)}} \\ &= \sup_{\varphi\neq 0}\frac{|a_\tau((I-P_{k-1})v,w_\varphi - v_{k-1})|}{\|\varphi\|_{L^2(\Omega)}} \\ &\le \sup_{\varphi\neq 0}\frac{C\|w_\varphi - v_{k-1}\|_{\tau,\alpha/2}\|(I-P_{k-1})v\|_{\tau,\alpha/2}}{\|\varphi\|_{L^2(\Omega)}}. \end{aligned}$$

We take the infimum over $v_{k-1} \in V_{k-1}$ to get

$$\|(I-P_{k-1})v\|_{L^2(\Omega)} \le C\|(I-P_{k-1})v\|_{\tau,\alpha/2}\left(\sup_{\varphi\neq 0}\left\{\frac{1}{\|\varphi\|_{L^2(\Omega)}}\inf_{v_{k-1}\in V_{k-1}}\|w_\varphi - v_{k-1}\|_{\tau,\alpha/2}\right\}\right).$$

Using the property of the interpolation operator I_h [10] and (3.3), we get

$$\begin{aligned} \inf_{v_{k-1}\in V_{k-1}}\|w_\varphi - v_{k-1}\|_{\tau,\alpha/2}^2 &\le \|w_\varphi - I_h(w_\varphi)\|_{\tau,\alpha/2}^2 \\ &= \tau^{-1}\|w_\varphi - I_h(w_\varphi)\|_{L^2(\Omega)}^2 + \|w_\varphi - I_h(w_\varphi)\|_{H^{\alpha/2}(\Omega)}^2 \\ &\le \tau^{-1}h^{2\alpha}\|w_\varphi\|_{H^\alpha(\Omega)}^2 + h^\alpha\|w_\varphi\|_{H^\alpha(\Omega)}^2 \le C_1h^\alpha\left(1+\tau^{-1}h^\alpha\right)\|w_\varphi\|_{H^\alpha(\Omega)}^2. \end{aligned}$$

According to the above equations and Assumption 4.1 of [2], there exists

$$\|(I-P_{k-1})v\|_{L^2(\Omega)} \le CC_2 h^{\alpha/2}\left(1+\tau^{-1}h^{\alpha}\right)^{1/2}\|(I-P_{k-1})v\|_{\tau,\alpha/2}.$$

The proof is completed. □

Lemma 3.7. *There exists a constant such that*

$$\|(I-P_{k-1})v\|_{\tau,\alpha/2} \le Ch^{\alpha/2}(1+\tau^{-1}h^{\alpha})^{1/2}|\!|\!|v|\!|\!|_{2,k,\tau} \quad \forall v\in V_k.$$

Proof. According to (3.4), (2.6), (2.4), (3.1) and Lemmas 3.5, 3.6

$$\begin{aligned}\|(I-P_{k-1})v\|^2_{\tau,\alpha/2} &\le C_1|\!|\!|(I-P_{k-1})v|\!|\!|^2_{1,k,\tau} = C_1(A_{k,\tau}(I-P_{k-1})v,(I-P_{k-1})v)_k\\ &= C_1a_\tau((I-P_{k-1})v,v) \le C_1\|(I-P_{k-1})v\|_{L^2(\Omega)}|\!|\!|v|\!|\!|_{2,k,\tau}\\ &\le C_1Ch^{\alpha/2}(1+\tau^{-1}h^{\alpha})^{1/2}\|(I-P_{k-1})v\|_{\tau,\alpha/2}|\!|\!|v|\!|\!|_{2,k,\tau}.\end{aligned}$$

The proof is completed. □

Definition 3.2. The error operator $E_k : V_k \to V_k$ is defined recursively by

$$E_1 = 0, \qquad E_k = K_k^m[I-(I-E_{k-1})P_{k-1}]K_k^m \quad \forall k\ge 1,$$

where $m = m_1 = m_2$ is given in Algorithm 1.

Lemma 3.8. *Let $z, g \in V_k$ satisfy $A_{k,\tau}z = g$ with the initial guess z_0. Then*

$$E_k(z-z_0) = z - \mathrm{MG}(k, z_0, g) \quad \forall k\ge 1.$$

Proof. The similar arguments can be performed as [5,10], we omit it here. □

Lemma 3.9. $a_\tau((I-K_k)K_k^{2m}v,v) \le \frac{1}{2m}a_\tau((I-K_k^{2m})v,v)$.

Proof. The similar arguments can be performed as [5,10], we omit it here. □

Lemma 3.10. *Let m be the number of smoothing steps and $\tau^{-1}h^{\alpha}\le C$ with $1<\alpha<2$. Then*

$$a_\tau(E_kv,v) \le \frac{C^*}{m+C^*}a_\tau(v,v) \quad \forall v\in V_k \tag{3.5}$$

where C^* is a positive constant independent of h and τ.

Proof. Let $\gamma = \frac{C^*}{m+C^*}$. We prove (3.5) by the mathematical induction. It obviously holds for $k = 1$ by Definition 3.2. Assume that

$$a_\tau(E_{k-1}v,v) \le \gamma a_\tau(v,v).$$

Next we prove that (3.5) holds. From Definition 3.2 and the above equation, it yields

$$a_\tau(E_kv,v) \le C_2(1-\gamma)\|(I-P_{k-1})K_k^mv\|^2_{\tau,\alpha/2} + \gamma a_\tau(K_k^mv,K_k^mv).$$

Table 1
MGM to solve the resulting system (2.5) with $x_L = 0$, $x_R = 32$, $T = 1$ and $\tau = T/N$, $h = x_R/M$, $N = M$.

N	$\alpha = 1.1$	Rate	Iter	CPU (s)	$\alpha = 1.7$	Rate	Iter	CPU (s)
2^7	2.7631e−03		13	1.29	3.2475e−03		11	0.86
2^8	6.9026e−04	2.0011	11	2.11	8.0166e−04	2.0183	9	1.79
2^9	1.7250e−04	2.0005	10	4.80	1.9810e−04	2.0168	8	4.07
2^{10}	4.2887e−05	2.0080	9	11.85	4.8927e−05	2.0175	6	8.64

According to Lemmas 3.7, 3.3 and 3.9, we get

$$\|(I - P_{k-1})K_k^m v\|^2_{\tau,\alpha/2} \leq C(1+\tau^{-1}h^\alpha)^2 \frac{1}{2m}(a_\tau(v,v) - a_\tau(K_k^m v, K_k^m v)).$$

Taking $C^* = \frac{CC_2(1+\tau^{-1}h^\alpha)^2}{2}$ and using the above equations, the desired results are obtained. □

Theorem 3.1. *Let m be the number of smoothing steps and $\tau^{-1}h^\alpha \leq C$ with $1 < \alpha < 2$. Then*

$$\|z - \mathrm{MG}(k, z_0, g)\|_{\tau,E} \leq \frac{C^*}{m + C^*}\|z - z_0\|_{\tau,E} \quad \forall z \in V_k,$$

where the time-dependent energy norm associated with (2.3) is defined by $\|z\|_{\tau,E} = \sqrt{a_\tau(z,z)}$.

Proof. Let μ_i be the eigenvalues of the operator E_k and φ_i be the corresponding eigenfunction satisfying the orthogonal relation $a_\tau(\varphi_i, \varphi_j) = \delta_{i,j}$. Using Lemma 3.10, we obtain $0 < \mu_1 \leq \mu_1 \cdots \mu_{n_k} \leq \gamma$, where $\gamma = \frac{C^*}{m+C^*}$ is given in (3.5). Let $v = \sum_{i=1}^{n_k} c_i \varphi_i$, we have

$$\|E_k v\|^2_{\tau,E} = a_\tau(E_k v, E_k v) = \sum_{i=1}^{n_k} c_i^2 \mu_i^2 \leq \gamma^2 a_\tau(v,v).$$

From Lemma 3.8 and the above equation, the desired results are obtained. □

4. Numerical results

We employ the V-cycle MGM described in Algorithm 1 to solve the resulting system. The stopping criterion is taken as $\frac{\|r^{(i)}\|}{\|r^{(0)}\|} < 10^{-10}$, where $r^{(i)}$ is the residual vector after i iterations; and the number of iterations $(m_1, m_2) = (1, 1)$ and $(\eta_{pre}, \eta_{post}) = (1/2, 1/2)$. The numerical errors are measured by the L_∞ norm, 'Rate' denotes the convergence orders. 'CPU' denotes the total CPU time in seconds (s) for solving the resulting discretized systems; and 'Iter' denotes the average number of iterations required to solve a general linear system $A_{k,\tau} z = g$ at each time level.

All numerical experiments are programmed in Matlab, and the computations are carried out on a PC with the configuration: Inter(R) Core (TM) i5-3470 CPU 3.20 GHZ and 8 GB RAM and a Windows 7 operating system.

Let us consider the time-dependent fractional problem (1.1) with $x_L < x < x_R$, $0 < t \leq T$. Take the exact solution of the equation as $u(x,t) = e^{-t}x^2(1 - x/x_R)^2$, then the corresponding initial and boundary conditions are, respectively, $u(x,0) = x^2(1-x/x_R)^2$ and $u(x_L,t) = u(x_R,t) = 0$; and the forcing function

$$f(x,t) = -e^{-t}x^2(1-x/x_R)^2 - e^{-t}\kappa_\alpha \left({}_{x_L}D_x^\alpha \left[x^2(1-x/x_R)^2\right] + {}_x D_{x_R}^\alpha \left[x^2(1-x/x_R)^2\right]\right).$$

From Table 1, we numerically confirm that the numerical scheme has second-order accuracy and the computational cost is almost $\mathcal{O}(N\log N)$ operations.

5. Conclusions

There are already some uniform convergence of V-cycle MGM to solve the time-dependent PDEs with a fixed time step $\tau > 0$. In this work, we introduce and analyse the fractional τ-norm, the convergence rate of the V-cycle MGM is strictly proved when $\tau \to 0$. We remark that the corresponding theory and numerical experiments can be extended to the time-fractional Feynman–Kac equation [9], the classical parabolic PDEs and the multidimensional cases.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (NSFC) under Grant No. 11601206. We would like to thank the anonymous referees for several suggestions and comments that led to much better results and an improved presentation.

References

[1] M.H. Chen, Y.T. Wang, X. Cheng, W.H. Deng, Second-order LOD multigrid method for multidimensional Riesz fractional diffusion equation, BIT 54 (2014) 623–647.
[2] V.J. Ervin, J.P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numer. Methods Partial Differential Equations 22 (2006) 558–576.
[3] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[4] R.E. Bank, T. Dupont, An optimal order process for solving finite element equations, Math. Comp. 153 (1981) 35–51.
[5] J.H. Bramble, J.E. Pasciak, New convergence estimates for multigrid algorithms, Math. Comp. 49 (1987) 311–329.
[6] J. Xu, L. Zikatanov, The method of alternating projections and the method of subspace corrections in Hilbert space, J. Amer. Math. Soc. 15 (2002) 573–597.
[7] R.Q. Jia, Applications of multigrid algorithms to finite difference schemes for elliptic equations with variable coefficients, SIAM J. Sci. Comput. 36 (2014) A1140–A1162.
[8] A. Aricò, M. Donatelli, S. Serra-Capizzano, V-cycle optimal convergence for certain (multilevel) structured linear systems, SIAM J. Matrix Anal. Appl. 26 (2004) 186–214.
[9] M.H. Chen, W.H. Deng, S. Serra-Capizzano, Uniform convergence of V-cycle multigrid algorithms for two-Dimensional fractional Feynman–Kac equation, J. Sci. Comput. 74 (2018) 1034–1059.
[10] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer, 2008.
[11] M. Bolten, M. Donatelli, T. Huckle, C. Kravvaritis, Generalized grid transfer operators for multigrid methods applied on Toeplitz matrices, BIT 55 (2015) 341–366.
[12] Y.J. Jiang, X.J. Xu, Multigrid methods for space fractional partial differential equations, J. Comput. Phys. 302 (2015) 374–392.
[13] Z.Q. Zhou, H.G. Wu, Finite element multigrid method for the boundary value problem of fractional advection dispersion equation, J. Appl. Math. (2013) 385463.
[14] L. Chen, R.H. Nochetto, E. Otárola, A.J. Salgado, Multilevel methods for nonuniformly elliptic operators and fractional diffusion, Math. Comp. 85 (2016) 2583–2607.
[15] H. Pang, H. Sun, Multigrid method for fractional diffusion equations, J. Comput. Phys. 231 (2012) 693–703.
[16] R.H. Chan, Q.S. Chang, H.W. Sun, Multigrid method for ill-conditioned symmetric Toeplitz systems, SIAM J. Sci. Comput. 19 (1998) 516–529.
[17] W.P. Bu, X.T. Liu, Y.F. Tang, J.Y. Yang, Finite element multigrid method for multi-term time fractional advection diffusion equations, Int. J. Model. Simul. Sci. Comput. 6 (2015) 1540001.
[18] W.L. Briggs, V.E. Henson, S.F. Mccormick, A Multigrid Tutorial, SIAM, 2000.
[19] A. Quarteroni, A. Valli, Numerical Approximation of Partial Differential Equations, Springer, 1994.
[20] X. Yue, W. Bu, S. Shu, M. Liu, S. Wang, Fully finite element adaptive algebraic multigrid method for time-space Caputo–Riesz fractional diffusion equations, Adv. Appl. Math. Mech. 10 (2018) 1103–1125.